On the propagation of shock waves through regions of non-uniform area or flow

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SUMMARY

This paper refers to the work of Moeckel (1952) on the interaction of an oblique shock wave with a shear layer in steady supersonic flow and the work of Chester (1955) and Chisnell (1957) on the propagation of a shock wave down a non-uniform tube. It is shown that their basic results can be obtained by the application of the following simple rule. The relevant equations of motion are first written in characteristic form. Then the rule is to apply the differential relation which must be satisfied by the flow quantities along a characteristic to the flow quantities just behind the shock wave. Together with the shock relations this rule determines the motion of the shock wave. The accuracy of the results for a wide range of problems and for all shock strengths is truly surprising.

The results are exactly the same as were found by the authors cited above. The derivation given here is simpler to perform (although the original methods were by no means involved) and of somewhat wider application, but the main reason for presenting this discussion is to try to throw further light on these remarkable results.

In discussing the underlying reasons for this rule, it is convenient to use the propagation in a non-uniform tube as a typical example, but applications to a number of problems are given later. A list of some of these appears at the beginning of the introductory section.

1. INTRODUCTION

There are various problems arising in the different branches of fluid dynamics which have as a main feature the interaction of a shock wave with a non-uniform region of some sort. Some examples are:

(i) the motion of a shock wave down a non-uniform channel or tube;
(ii) the propagation of a shock normally through a plane distribution of density, pressure, etc.;
(iii) converging cylindrical and spherical shocks (including non-uniform states ahead of the shock such as occur in magnetohydrodynamics);

(iv) the interaction of an oblique shock with a shear layer in steady supersonic flow;

(v) the shape and strength of the shock inside the entry of an axi-symmetrical supersonic duct;

(vi) the propagation of a bore in water of non-uniform depth;

(vii) the motion of kinematic shock waves in traffic flow and flood waves when the flow conditions downstream are non-uniform.

We shall ultimately consider all of these problems, but in the general discussion we may refer to (i) as a typical example. It should be stated at once that only the one-dimensional (hydraulic) formulation is intended in (i), i.e. all quantities are averaged over the cross-sectional area and become functions of the time and distance down the tube only.

Of course, even in uniform media, the problem of shock propagation cannot be solved analytically in general and numerical methods have to be used. But to study some of the main effects of the interaction we can consider cases in which changes in the speed and strength of the shock are caused entirely by the non-uniform region. Thus, for the propagation down a non-uniform tube we suppose that to the left ($x < 0$) of some section $x = 0$, the cross-sectional area $A(x)$ is constant and the shock wave initially moves in this constant part of the tube travelling with uniform speed. Taking the usual model, we can suppose that the shock wave is caused by a piston moving with constant speed; the piston is always taken far enough to the left for the reflections from it not to come back and interfere with the basic interaction. The analogous suppositions for the other problems listed above are obvious except perhaps in the case of (iii) and (v). But the converging shocks of (iii) can be treated as special cases of the non-uniform tube. The cylindrical shock is obtained by choosing a wedge-shaped channel with $A(x) \propto (x_0 - x)$ and the spherical shock by taking a cone-shaped tube with $A(x) \propto (x_0 - x)^2$. In this formulation we can still suppose there is a uniform section with A constant before the varying part. However, Guderley's similarity solutions for converging shocks (Guderley 1942) will be discussed below and they do not correspond exactly to these initial conditions. But converging cylindrical shocks are known to be insensitive to the details of the initial conditions. For example, Payne's numerical calculations (Payne 1957) agree closely with Guderley's solution although the initial conditions are quite different (in fact, they are close to the ones proposed above). The same thing will be found here, with similar results for problem (v).

Apart from the special features of (iii) and (v) the initial conditions adopted above will not be exactly fulfilled in practice in the other cases; even without the non-uniformities in the medium, the shock may be varying due to changes in the flow conditions behind it (such as variations in the speed of the effective piston in the model mentioned above). But in many cases, the scale of this variation will be much greater than that of the interaction and so will have a negligible effect on the 'local' behaviour in the non-uniform region, The considerations here arise from the work of Moeckel (1952), Chester (1954), and Chisnell (1955, 1957). Moeckel studied problem (iv) by the following method. The shear layer is represented by a set of parallel surfaces at which small discontinuities in velocity, pressure, density, etc., occur. The interaction of an oblique shock with any of these can be solved to find the resulting change in the strength of the shock in terms of the change in the incident flow. The successive interactions are not independent, but by ignoring the dependence between them comparatively simple laws are obtained for the change in strength of the shock as it progresses through the shear layer. With this basic result Moeckel goes on to discuss refinements. Chisnell uses essentially the same methods to study problems (i) and (ii); on the whole, it is more convenient to consider these as far as basic ideas are concerned. In the first of them the elementary interaction is between the shock and a small change δA in cross-sectional area; the resulting change δM in the Mach number M of the shock wave is given by the formula

$$\frac{\delta A}{A} = -\frac{2M\delta M}{(M^2-1)K(M)}, \tag{1}$$

where $K(M)$ is a slowly varying function which starts at 0·5 for weak shocks, $M = 1$, and tends to 0·394 (for $\gamma = 1{\cdot}4$) as $M \to \infty$. Chisnell represents the tube by a series of such small changes and for a first approximation neglects the interference between successive interactions. This amounts to integrating (1) to get M as a function of A. In the limit as $M \to 1$, this gives

$$M-1 \propto A^{-\frac{1}{4}}, \tag{2}$$

which is the law of geometrical acoustics; in the limit as $M \to \infty$, it gives

$$M \propto A^{-\frac{1}{2}K_\infty}, \tag{3}$$

where $K_\infty = 0{\cdot}394$ for $\gamma = 1{\cdot}4$, as noted above. Now (2) is well established for weak shocks* and Chisnell checked (3) by applying his formula to converging cylindrical and spherical shocks and comparing with Guderley's exact solutions for these cases. The cylindrical shock corresponds to a wedge-shaped tube with A proportional to the distance from the axis and the spherical one corresponds to a conical tube with A proportional to the square of the distance from the centre. Thus, according to (3), the exponents in the formula for M as a function of distance are $\frac{1}{2}K_\infty$ and K_∞ respectively. The comparison with the Guderley values (as calculated by Butler (1954)) is shown in the following table:

γ	Cylindrical shock		Spherical shock	
	Chisnell	Guderley	Chisnell	Guderley
6/5	0·163112	0·161220	0·326223	0·320752
7/5	0·197070	0·197294	0·394141	0·394364
5/3	0·225425	0·226054	0·452108	0·452692

* In the present circumstances this is true; there are important differences between weak shocks and acoustic pulses in general.

Comment on these is unnecessary! Payne (1957) determined the motion of converging cylindrical shocks numerically for a whole range of shock strengths and found that in all cases Chisnell's law was extremely accurate. Chisnell went on to find a higher approximation by including re-reflected waves and the corrections turned out to be of the expected small magnitude. In his approach the high accuracy of the first approximations appears to some extent as a coincidence.

Chester's contribution was on different lines. He considered problem (i) for the case in which the cross-sectional area $A(x)$ remains close to some mean value. He developed a theory for the small perturbations in the flow behind the shock and solved the linearized equations exactly. The formula (1) was first obtained by Chester in this way. Although Chester's work is more restricted than Chisnell's, it adds greatly to an understanding of these results; it is discussed in detail in the next section.

Now, the purpose of the present paper is to point out that the basic formulae obtained by these authors can be derived in the following significant way. First the appropriate equations of motion for the flow are written in characteristic form. For example, for a non-uniform tube we have

$$dp + \rho a\, du + \frac{\rho a^2 u}{u+a}\frac{dA}{A} = 0 \tag{4}$$

on a positive characteristic $dx/dt = u+a$, where p, ρ, a, u denote the pressure, density, sound speed, particle velocity, respectively. Then, the characteristic relation is applied (quite illogically it may seem) to the flow quantities at the shock wave. But these quantities are all known in terms of the shock strength from the Rankine–Hugoniot shock relations. Thus on substitution in the characteristic equation, an equation for the variation of the shock strength is obtained. For example, in (4) we substitute the expressions given by the shock relations for p, ρ, a, u in terms of the Mach number of the shock. This gives a first-order equation for M as a function of A which can be integrated immediately. It is exactly Chisnell's formula, the differential equation for $M(A)$ being identical with (1). The constant of integration is fixed from the initial strength of the shock in the straight portion of the tube. When the area A is constant the variations in shock strength caused by non-uniform conditions ahead of the shock come in only through the shock relations.

Even for quite complicated equations it is usually a simple matter to write down the characteristic equation corresponding to (4). In addition only the shock relations are required—and these are needed in any approach.

Once this derivation has been noticed it is easy to see why it agrees with the previous methods. Full details are given in the subsequent sections. New questions also arise. For instance, this new approach is found to have a close connection with Butler's method for calculating the Guderley solutions for converging shocks. Again the method is related in some way with 'shock-expansion theory'. These various topics and additional applications are discussed below; the section headings are self-explanatory.

In concluding these general remarks it should be said that the discussions in this paper still fall short of a full understanding of all the questions involved; it is still not completely clear to what extent the unexpected accuracy of the results in some cases should be ascribed to coincidence.

2. Detailed discussion of the method

As mentioned already, we shall consider first the typical example of the propagation of a shock down a tube of variable area $A(x)$ containing gas originally at rest with constant pressure p_0 and density ρ_0. We suppose that $A(x)$ takes a constant value A_1 in $x < 0$ and that the shock is initially

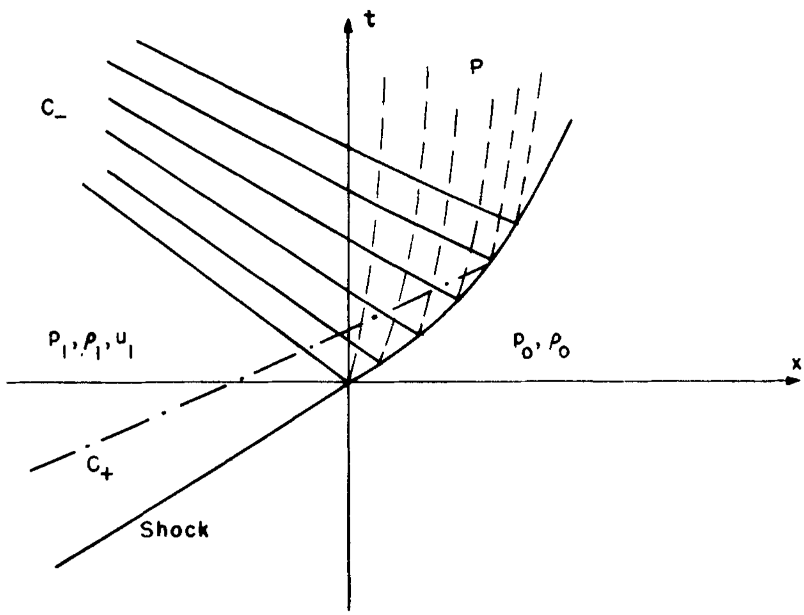

Figure 1. The (x, t)-plane for the interaction of a shock wave with a non-uniform region.

moving with constant velocity U_1 in this region. The flow quantities p_1, ρ_1, a_1, u_1 behind the shock are determined in terms of U_1 by the shock conditions. Conversely, given u_1 (the velocity of a piston maintaining the motion for example) the shock velocity U_1 and the other quantities can be found. When the shock reaches $x = 0$, disturbances are propagated back into the uniform flow region as represented in figure 1, and the future motion of the shock is modified. This reflected disturbance propagates along negative characteristics labelled C_- in the figure; in addition, entropy changes are carried along the particle paths labelled P. From the physical point of view the positive characteristics (one labelled as C_+ is shown in figure 1) play a subsidiary role. We may note that in $x < 0$ the C_- form a simple wave, i.e. they are all straight and $2a/(\gamma-1)+u = 2a_1/(\gamma-1)+u_1$

throughout. (We assume that if a reflected shock wave is formed it does so outside the region under consideration.)

The equations of motion are

$$\left.\begin{aligned} \rho_t + u\rho_x + \rho\left(u_x + \frac{uA'}{A}\right) &= 0, \\ u_t + uu_x + \frac{1}{\rho}p_x &= 0, \\ p_t + up_x - a^2(\rho_t + u\rho_x) &= 0, \end{aligned}\right\} \tag{5}$$

where $a^2 = \gamma p/\rho$. The characteristic form (i.e. the form for which each equation contains derivatives in only one direction in the (x, t)-plane) is

$$\left.\begin{aligned} \left(\frac{p_t}{u \pm a} + p_x\right) \pm \rho a\left(\frac{u_t}{u \pm a} + u_x\right) + \frac{\rho a^2 u A'}{(u \pm a)A} &= 0, \\ \left(\frac{p_t}{u} + p_x\right) - a^2\left(\frac{\rho_t}{u} + \rho_x\right) &= 0. \end{aligned}\right\} \tag{6}$$

We can also write these as

$$\left.\begin{aligned} dp + \rho a\, du + \frac{\rho a^2 u}{u+a}\frac{dA}{A} &= 0 \quad \text{on } C_+ : \quad \frac{dx}{dt} = u + a, \\ dp - \rho a\, du + \frac{\rho a^2 u}{u-a}\frac{dA}{A} &= 0 \quad \text{on } C_- : \quad \frac{dx}{dt} = u - a, \\ dp - a^2\, d\rho &= 0 \quad \text{on } P : \quad \frac{dx}{dt} = u. \end{aligned}\right\} \tag{7}$$

The rule is to apply the first of these (valid along a C_+) to the shock wave. From the shock conditions

$$\left.\begin{aligned} p &= \rho_0 a_0^2\left\{\frac{2}{\gamma+1}M^2 - \frac{\gamma-1}{\gamma(\gamma+1)}\right\}, \\ \rho &= \rho_0\frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}, \\ u &= a_0\frac{2}{\gamma+1}\left(M - \frac{1}{M}\right), \end{aligned}\right\} \tag{8}$$

where the shock velocity $U = a_0 M$. Substituting these expressions into the first of (7), we have

$$\frac{2M}{(M^2-1)}\frac{dM}{K(M)} + \frac{dA}{A} = 0, \tag{9}$$

where

$$K(M) = 2\left[\left(1 + \frac{2}{\gamma+1}\frac{1-\mu^2}{\mu}\right)(2\mu + 1 + M^{-2})\right]^{-1}, \tag{10}$$

$$\mu^2 = \left(\frac{U-u}{a}\right)^2 = \frac{(\gamma-1)M^2+2}{2\gamma M^2 - (\gamma-1)}.$$

The law of propagation of the shock is given by the function $M(A)$ which satisfies (9).

As $M \to 1$, $\mu \to 1$ and $K \to 0{\cdot}5$; hence, the solution of (9) is

$$M-1 \propto A^{-\frac{1}{2}}.$$

As $M \to \infty$, $\mu^2 \to (\gamma-1)/2\gamma$ and

$$K \to K_\infty = 2\left[\left\{1+\sqrt{\left(\frac{2}{\gamma(\gamma-1)}\right)}\right\}\left\{1+\sqrt{\left(\frac{2(\gamma-1)}{\gamma}\right)}\right\}\right]^{-1}$$

$$= 0{\cdot}394 \quad \text{for } \gamma = 1{\cdot}4; \tag{11}$$

hence,

$$M \propto A^{-\frac{1}{2}K_\infty}.$$

The corresponding laws for the variations of p, ρ, u are then found from (8).

We must now investigate why this rule works so well. For some reason

$$dp+\rho a\, du+\frac{\rho a^2 u}{u+a}\frac{dA}{A}$$

is almost zero along the shock. This means that

$$\left(\frac{p_t}{U}+p_x\right)+\rho a\left(\frac{u_t}{U}+u_x\right)+\frac{\rho a^2 u}{u+a}\frac{A'}{A}$$

is very small at the shock. But, by making use of the first of equations (6) (which holds everywhere), we conclude that

$$\left(\frac{1}{U}-\frac{1}{u+a}\right)(p_t+\rho a u_t) \tag{12}$$

is very small at the shock. Now the obvious first thought as to why the rule works is that the characteristic C_+ follows along behind the shock very close to it (see figure 1), and so the relation valid along the C_+ is a good approximation along the shock. This would appeal to a supposed smallness of the first factor in (12), but although $(u+a-U)/U$ is zero for $M = 1$, it tends to $[\sqrt{\{2\gamma(\gamma-1)\}}-(\gamma-1)]/(\gamma+1) = 0{\cdot}274$ (for $\gamma = 1{\cdot}4$) as $M \to \infty$, which is at least a hundred times the error found in the results. *In fact, it is the smallness of the second factor in* (12) *which leads to the high accuracy.*

In Chester's small perturbation theory it will be shown that $p_t+\rho a u_t$ is zero at the shock; in fact $p_t+\rho a u_t$ is zero everywhere and the above result is the correct answer in that theory. However, when finite changes in A are considered, as in Chisnell's work on the cylindrical shock, it is not clear why this should be so. From the numerical calculations it is observed that $p_t+\rho a u_t$ is very small, but all the analytical approaches considered so far work on expansions assuming essentially that the first factor in (12) is small. Then the high accuracy of the lowest order results comes out as something of a coincidence. However, these approaches are very relevant to Butler's calculations and some comments are made in the next section.

First we consider Chester's small perturbation theory. It is clear at the outset that if p, ρ, a, u remain close to the original values p_1, ρ_1, a_1, u_1 respectively, then in the linearized theory we have

$$dp+\rho_1 a_1\, du+\frac{\rho_1 a_1^2 u_1}{u_1+a_1}\frac{dA}{A_1}=0$$

on each C_+. Since each C_+ starts in the region where $p=p_1$ etc., it follows that

$$(p-p_1)+\rho_1 a_1(u-u_1)+\frac{\rho_1 a_1^2 u_1}{u_1+a_1}\frac{A(x)-A_1}{A_1}=0 \tag{13}$$

throughout the flow and in particular at the shock. On substituting the shock relations the result derived above is obtained. Also, differentiating (13) with respect to t, we see that $p_t+\rho_1 a_1 u_t=0$ everywhere. The relation (13) must also apply across one of Chisnell's interactions and this explains why the present method gives the same answer as Chisnell's.

It is interesting to look at Chester's full solution. The equations (5) may be linearized in the straightforward way* and the general solution is found to be

$$p-p_1=-\frac{\rho_1 a_1^2 u_1^2}{u_1^2-a_1^2}\frac{A(x)-A_1}{A_1}+F(x-\{u_1+a_1\}t)+G(x-\{u_1-a_1\}t), \tag{14}$$

$$u-u_1=\frac{a_1^2 u_1}{u_1^2-a_1^2}\frac{A(x)-A_1}{A_1}+\frac{1}{\rho_1 a_1}F(x-\{u_1+a_1\}t)-\frac{1}{\rho_1 a_1}G(x-\{u_1-a_1\}t), \tag{15}$$

$$\rho-\rho_1=\frac{p-p_1}{a_1^2}+H(x-u_1 t), \tag{16}$$

where F, G and H are arbitrary functions. This solution shows very clearly how the disturbances are made up of the four contributions; first, terms directly from the area change, then disturbances propagating on

$$x-\{u_1+a_1\}t=\text{constant } (C_+), \qquad x-\{u_1-a_1\}t=\text{constant } (C_-)$$

and

$$x-u_1 t=\text{constant } (P).$$

The entropy changes are represented by the function H and it is interesting to note that p and u do not depend directly on them. The shock conditions provide two relations between p, ρ and u; these boundary conditions serve to determine the functions G and H. The other function F is determined by the flow in $x<0$ and in the present case $F=0$†. With $F=0$ it is observed from (14) and (15) that $p_t+\rho_1 a_1 u_t=0$.

* Actually Chester gives a more thorough treatment: he starts with the full three-dimensional equations of motion and carries through in detail an averaging process which leads to the hydraulic theory.

† Contributions to F arise if the piston motion in the straight portion of the tube is not uniform.

In general, $p_t + \rho_1 a_1 u_t = -2(u_1 + a_1)F(x - \{u_1 + a_1\}t)$ so that $p_t + \rho_1 a_1 u_t$ is constant along a positive characteristic C_+; the t-derivatives knock out the terms depending directly on A and this particular combination gets rid of the G. In the general problem when the disturbances are not small, we may expect that in any small region the small perturbation theory holds for variations about some local values. Then for each local region, $p_t + \rho a u_t$ remains constant on a C_+. This leads naturally to the suggestion that $p_t + \rho a u_t$ varies very little along a C_+ even when large regions and finite changes are involved. If this were so we could argue that all C_+ characteristics start in $x < 0$ and $p_t + \rho a u_t = 0$ there; hence, $p_t + \rho a u_t = 0$ along each C_+ and in particular where they meet the shock. Roughly speaking, $p_t + \rho a u_t$ would play the role of the Riemann invariant in simple wave theory. Indeed in a negative simple wave $p_t + \rho a u_t$ is constant. Thus, for example, in the simple wave formed by the C_- in $x < 0$ in figure 1, $p_t + \rho a u_t$ vanishes, *even though the total changes in p and u may not be small.* This provides useful support for the contention that $p_t + \rho a u_t$ is small. However, in the general case the possibility that small changes may accumulate over a large region cannot be overlooked. Presumably, the result would only hold for flows which are slowly varying in some appropriate sense. But a precise definition of 'slowly-varying' which would include cylindrical and spherical shocks, whose strengths ultimately become infinite, is far from clear. The author has so far been unable to make further headway in this direction. The expression for the rate of change of $p_t + \rho a u_t$ on a C_+ is quadratic in the first derivatives of the flow quantities and so is of smaller order when these derivatives are small, but the expression is quite complicated and does not seem to suggest anything else significant. It is true that it could be used to find a correction term in going to a next higher approximation. But the smallness of the correction will only confirm the accuracy of the lowest order approximation without throwing further light on the reasons for it.

This can be said of all the other approaches tried so far. However, one of them is quite systematic and gives a series of successive approximations with (9) as the first approximation. As a mathematical procedure for finding the answer it is satisfactory, but it does not explain why the first approximation is so good. This is because it works essentially on the first factor in (12) instead of the much smaller second factor. Thus one feels the method cannot be the right one. It is, however, closely related to Butler's calculations for the converging cylindrical shocks and so is worth describing. This is done in the next section for those particular problems.

3. Converging cylindrical and spherical shocks

We must first consider Guderley's solution briefly. Guderley (1942) studied the limiting case of a very strong shock and found that there exists a similarity solution representing a converging cylindrical or spherical shock. For the cylindrical case we take equations (5) with $A \propto (x_0 - x)$

corresponding to a wedge-shaped channel. Guderley's similarity solution takes the form

$$u = \frac{x_0 - x}{t} f(\xi), \qquad p = \rho_0 \frac{(x_0 - x)^2}{t^2} g(\xi), \qquad \rho = \rho_0 h(\xi), \tag{17}$$

where $\xi = (x_0 - x)/t^\alpha$ and α is an exponent to be determined. The shock is a line ξ = constant = ξ_0, say, so that its motion is given by the law $x_0 - x = \xi_0 t^\alpha$. For example the shock velocity varies like $(x_0 - x)^{-n}$ where $n = (1-\alpha)/\alpha$. When the expressions (17) are substituted in (5), ordinary differential equations are obtained for f, g, h and the solution satisfying the shock conditions at $\xi = \xi_0$ is required. The exponent α is determined in a rather strange way; there is only one value of α which gives a solution free of unrealistic singularities in the flow. For other values of α a fold occurs

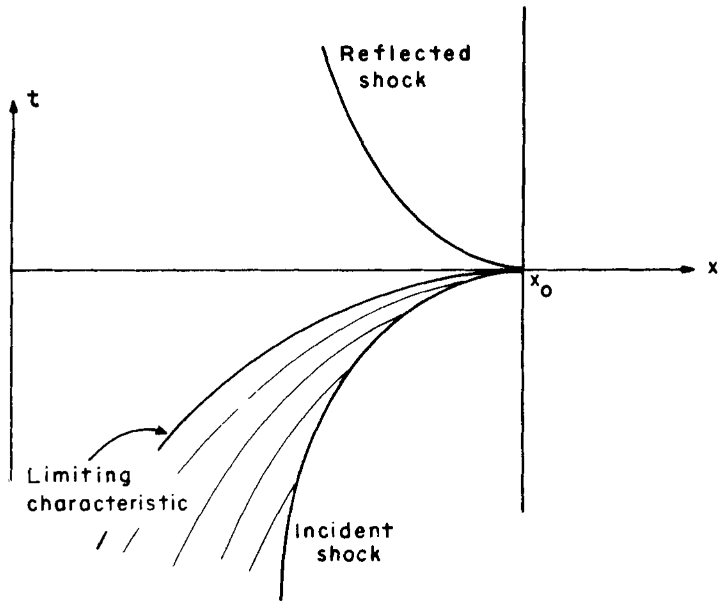

Figure 2. The (x, t)-plane for a converging cylindrical shock wave.

in the (x, t)-plane and the solution is no longer single-valued. This fold would be formed along a particular line $\xi = \xi_1$ which turns out to be the characteristic passing through the centre $x_0 - x = 0$ (see figure 2). To avoid the singularity the solution must have certain special properties at $\xi = \xi_1$ and these essentially determine α. One of Butler's suggestions (Butler 1954) for calculating the solution is to start with a series expansion near $\xi = \xi_1$, thus ensuring regularity there, *then* the shock conditions at $\xi = \xi_0$ are sufficient to complete the solution and determine α.

We now consider this sort of approach directly in the physical plane without using the similarity solution. This is independent of the assumption of a strong shock and can be extended directly to other problems. Let $p^{(0)}(x)$, $u^{(0)}(x)$, etc., denote the values of flow quantities on the limiting characteristic. Then the characteristic is given by $\tau = 0$ where

$$\tau = t - \int_{x_0}^{x} \frac{dx}{u^{(0)} + a^{(0)}}. \tag{18}$$

Next, all the flow quantities are expanded in power series in τ:

$$\begin{aligned} p(x,t) &= p^{(0)}(x) + \tau p^{(1)}(x) + \tau^2 p^{(2)}(x) + \ldots, \\ u(x,t) &= u^{(0)}(x) + \tau u^{(1)}(x) + \tau^2 u^{(2)}(x) + \ldots \text{ etc.} \end{aligned} \tag{19}$$

If we substitute these in the equations of motion and equate coefficients of τ to zero, we get the characteristic relation

$$\frac{dp^{(0)}}{dx} + \rho^{(0)} a^{(0)} \frac{du^{(0)}}{dx} + \frac{\rho^{(0)} a^{(0)2} u^{(0)}}{u^{(0)} + a^{(0)}} \frac{1}{(x_0 - x)} = 0 \tag{20}$$

connecting the lowest order terms, and a set of equations for determining the higher order coefficients $p^{(1)}(x)$, $p^{(2)}(x)$, etc., in terms of $p^{(0)}$, $\rho^{(0)}$, $u^{(0)}$. (This is in accordance with the general theory of characteristics.) Now, the boundary conditions (8) at the shock require

$$\left.\begin{aligned} p^{(0)}(x) + T(x) p^{(1)}(x) + \ldots &= \rho_0 a_0^2 \left\{ \frac{2}{\gamma+1} M^2 - \frac{\gamma-1}{\gamma(\gamma+1)} \right\}, \\ \rho^{(0)}(x) + T(x) \rho^{(1)}(x) + \ldots &= \rho_0 \frac{(\gamma+1) M^2}{(\gamma-1) M^2 + 2}, \\ u^{(0)}(x) + T(x) u^{(1)}(x) + \ldots &= a_0 \frac{2}{\gamma+1} \left(M - \frac{1}{M} \right), \end{aligned}\right\} \tag{21}$$

where $T(x)$ is the value of τ at the shock, i.e.

$$T(x) = \int_{x_0}^{x} \left\{ \frac{1}{U} - \frac{1}{u^{(0)} + a^{(0)}} \right\} dx. \tag{22}$$

For a first approximation, only the zero order terms, i.e. the values on the characteristic $\tau = 0$ are retained on the left of (21). Clearly, then, on substitution into (20), we have exactly the simple rule of this paper. From this first approximation for $p^{(0)}$, $\rho^{(0)}$, $u^{(0)}$, the coefficients $p^{(1)}$, $\rho^{(1)}$, $u^{(1)}$ can be found and so for a second approximation two terms can be retained on the left of each of the equations in (21). This gives improved values of $p^{(0)}$, $\rho^{(0)}$, $u^{(0)}$ in terms of M, which (on substitution in (20)) leads to an improved expression for $M(x)$ and so on. Of course the improved function $M(x)$ differs very little from the first approximation. It is an unexpected bonus in this approach. For, the whole approach is based on an expansion in powers of $(u+a-U)/U$ (see (22)) which is not so very small. In fact the values of $p^{(0)}$, $\rho^{(0)}$, $u^{(0)}$ change by the expected much larger amounts between the first and second approximations. For a strong shock, these quantities are proportional to powers of $(x_0 - x)$, and for $\gamma = 1{\cdot}4$ the coefficients in the two approximations are:

	First approximation	Second approximation
$p^{(0)}$	0·833	0·967
$\rho^{(0)}$	6·000	7·420
$u^{(0)}$	0·833	0·784

Putting it slightly differently, the first approximation takes just the values at the shock and the second approximation gives improved values at the characteristic. The reason for the small change in the law of propagation for the shock is that the correction to the exponent is proportional to

$$p^{(1)} + \rho^{(0)} a^{(0)} u^{(1)}$$

and this is very small. In fact this quantity is the value of $p_t + \rho a u_t$ on the characteristic! Thus, the previous comments are vindicated.

4. Connection with shock expansion theory

Shock expansion theory treats quite a different aspect of shock propagation but there are one or two points worth mentioning. This method concerns propagation down a uniform tube when the effective piston motion varies. (Its practical importance lies in the applications to the analogous problem of supersonic aerofoils.) When the shock is weak, the flow is approximately a simple wave on the C_+ characteristics with the Riemann invariant α given by

$$\alpha \equiv \frac{2a}{\gamma - 1} - u = \frac{2a_0}{\gamma - 1}$$

and the entropy S equal to the undisturbed value S_0. If the shock is not weak, a surprisingly good approximation to the pressures on the piston is obtained by assuming that the flow is a simple wave with

$$\frac{2a}{\gamma - 1} - u = \frac{2a_1}{\gamma - 1} - u_1, \qquad S = S_1$$

where the subscripts indicate values just behind the shock initially (these are easily found from the shock relations and the initial velocity of the piston). This is shock-expansion theory. The formula actually used for the pressures on the piston is

$$dp - \rho a \, du = 0, \tag{23}$$

because this holds for the simple wave. Since $S = S_1$ is exactly true on the piston, ρ and a can be expressed in terms of p, and (23) then determines the pressure variations in terms of the piston motion. Now, (23) is related to the fact that the characteristic equation appropriate to the C_- is

$$\{p_t + (u-a)p_x\} - \rho a\{u_t + (u-a)u_x\} = 0,$$

(for a uniform tube). Thus if

$$p_t - \rho a u_t = 0, \tag{24}$$

then (23) follows. This is the same condition as the one used in the work on non-uniform tubes but with the opposite sign. The sign change is because the main disturbance is on the C_+ characteristics here, but for the non-uniform problems (with a uniform piston motion) it is on the C_-. The same thing can be seen from Chester's solution (14), (15), (16); we have

$$p_t - \rho_1 a_1 u_t = -(u_1 - a_1)G'(x - \{u_1 - a_1\}t), \tag{25}$$

so that the reflected disturbance given by G is ignored in shock-expansion theory. In our discussion of non-uniform regions G is one of the main terms. We may also remark that shock-expansion theory makes this additional approximation of neglecting G even in the small perturbation theory. The treatment of non-uniform regions is exact in that theory. For further details of shock-expansion theory see Eggars, Savin & Syvertson (1955) and Mahony (1955).

5. Kinematic shocks

Analogous and much simpler shock problems arise in the subject called 'kinematic waves' by Lighthill & Whitham (1955), and one might hope that discussion of these easier cases would give further insight into the more complicated cases. In fact it does not: *the rule is correct but trivial.* We consider, for example, kinematic waves in traffic flow on crowded roads. The density of the traffic, k, and the rate of flow across any section, q, must satisfy the continuity equation

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{26}$$

In addition q will be some function of k, since the average speed of the cars (q/k) will fall as k increases. But if the road conditions are non-uniform, this relation will also include x. A shock wave, i.e. a discontinuous jump from conditions (q_1, k_1) to (q_2, k_2), travels with speed

$$U = \frac{q_2 - q_1}{k_2 - k_1}; \tag{27}$$

this is the only shock condition. The problem is to find out how such a shock, initially travelling with constant speed along a uniform stretch of road, varies when it reaches a non-uniform section specified by

$$k = f(q, x). \tag{28}$$

The solution is obvious: the shock continues to separate regions in which q takes the constant values q_1 and q_2 respectively. The corresponding values of k are functions of x given by (28), and (26) is satisfied. The shock velocity varies according to (27).

Now this solution is exactly the one given by the rule of this paper. The characteristic relation from (26) and (27) is

$$\frac{\partial f(q,x)}{\partial q}\frac{\partial q}{\partial t}+\frac{\partial q}{\partial x}=0 \quad \text{or } dq=0.$$

Hence the rule takes q = constant along the shock, i.e. q_2 is constant. Then since q_1 is given to be constant ahead of the shock, we have exactly the above solution. The rule is correct for this type of wave motion but is so trivial that it does not help with the more complicated cases.

Further applications

6. Propagation through a stratified layer

Suppose we have a stratified layer with the equilibrium values of pressure, density, etc. depending on x. In general we can suppose that the layer is maintained by a body force F (which could also vary with x). Then the equations of motion are

$$\left.\begin{aligned} \rho_t+u\rho_x+\rho u_x &= 0,\\ u_t+uu_x+\frac{1}{\rho}p_x &= F,\\ p_t+up_x-a^2(\rho_t+u\rho_x) &= 0. \end{aligned}\right\} \tag{29}$$

In equilibrium $p=p_0(x)$, $\rho=\rho_0(x)$ where

$$\frac{1}{\rho_0}\frac{dp_0}{dx}=F. \tag{30}$$

The determination of p_0 in terms of ρ_0 is completed by giving the equilibrium entropy distribution.

Chisnell (1955) considered this problem in the special case $F=0$, p_0 = constant (ρ_0 not constant) by replacing the continuous density distribution by a piecewise constant function. The successive layers, with ρ constant in each, can remain in equilibrium because the pressure is the same in each. But for varying p_0 it is not quite obvious how to proceed by that method, since jumps in p_0 cannot really be allowed at the interfaces; there is no force to balance the pressure difference (F is a *body* force). For the present rule, there is no difficulty. The appropriate characteristic relation is

$$\left(\frac{p_t}{u+a}+p_x\right)+\rho a\left(\frac{u_t}{u+a}+u_x\right)-\frac{\rho a}{u+a}F=0$$

or

$$dp+\rho a\,du-\frac{\rho a}{u+a}F\,dx=0. \tag{31}$$

The final step is to substitute the shock relations to determine the variation of M with x. In general the resulting first-order equation $M(x)$ will require numerical integration. However, in the special case of very strong shocks, (8) simplifies to

$$p=\frac{2}{\gamma+1}\rho_0U^2,\qquad \rho=\frac{\gamma+1}{\gamma-1}\rho_0,\qquad u=\frac{2}{\gamma+1}U, \tag{32}$$

and we see that the third term in (31) can be neglected (since U is very large). Thus the contribution of F is made indirectly through its control of the equilibrium density distribution $\rho_0(x)$. With these approximations the law of propagation of the shock is

$$U \propto \rho_0^{-\beta}, \qquad p \propto \rho_0^{1-2\beta},$$

where

$$\beta = \left\{2+\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)}\right\}^{-1}. \tag{33}$$

For $\gamma = 1{\cdot}4$, $\beta = 0{\cdot}2158$.

A specific example where this law may be useful is in finding the effect of the non-uniformities in the atmosphere on explosion blasts. In that case, F is the acceleration due to gravity, which is $-g$ if x is measured upwards. If we consider an atmospheric layer in adiabatic equilibrium, i.e. $p_0 = \kappa\rho_0^\gamma$, we have (from (30)):

$$\frac{\gamma\kappa\rho_0^{\gamma-1}}{\gamma-1} = C - gx, \tag{34}$$

where C is a constant.

7. Converging cylindrical shock waves in magnetohydrodynamics

This is a problem of some current interest which we can consider as a useful example where both changes in area and equilibrium density distribution occur. In such cases similarity solutions of Guderley's type are limited to power law distributions of density (as well as to very strong shocks) but the methods of this paper can be applied to the general solution. The investigation will be based on the Lundquist equations of magnetohydrodynamics (Lundquist 1952), and in particular the shock relations, etc., are all taken from a report by Friedrichs (1955).

In terms of the velocity vector $\mathbf{u}$, the magnetic field vector $\mathbf{H}$, pressure p and density ρ, the equations are

$$\frac{\partial \mathbf{H}}{\partial t} + \nabla \times (\mathbf{H}\times\mathbf{u}) = 0, \tag{35}$$

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho\mathbf{u}) = 0, \tag{36}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} . \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\mathbf{H}\times(\nabla\times\mathbf{H}) = 0, \tag{37}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} . \nabla p - a^2\left(\frac{\partial \rho}{\partial t} + \mathbf{u} . \nabla p\right) = 0, \tag{38}$$

where μ is the permeability. The Maxwell equation $\nabla . \mathbf{H} = 0$ is essentially included in (35) since the divergence of that equation gives $\partial(\nabla . \mathbf{H})/\partial t = 0$.

For flows with cylindrical symmetry it is assumed that $\mathbf{u}$ is radial and that all flow quantities are functions of the radial distance r and the time t. The transverse and axial components H_θ, H_z of $\mathbf{H}$ need not vanish, however. Indeed they are of primary importance since it is easily shown that the radial component H_r must vanish or the solution becomes trivial. For, if $H_r \neq 0$, the condition that the θ and z components of $\mathbf{H} \times (\nabla \times \mathbf{H})$ must vanish in (37) requires that rH_θ and H_z be independent of r. In turn these lead to artificial forms for u and the only feasible case turns out to be $H_\theta = H_z = 0$. But then the flow problem is independent of the magnetic field. Thus, only the case $H_r = 0$ is considered.

With these conditions of symmetry, equations (35)–(38) reduce to

$$\frac{\partial H_\theta}{\partial t} + \frac{\partial}{\partial r}(H_\theta u) = 0,$$

$$\frac{\partial H_z}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rH_z u) = 0,$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \rho\frac{u}{r} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{2}\frac{\mu}{\rho}\frac{\partial}{\partial r}(H_\theta^2 + H_z^2) + \frac{\mu}{\rho}\frac{H_\theta^2}{r} = 0,$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} - a^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right) = 0.$$

In the equilibrium state ahead of the shock, these equations reduce to

$$p_0 + \tfrac{1}{2}\mu(H_{\theta 0}^2 + H_{z0}^2) + \mu\int\frac{H_{\theta 0}^2}{r}\,dr = \text{constant}, \tag{39}$$

which determines the pressure distribution in terms of the magnetic field. (Subscripts zero are used for equilibrium values.) The density distribution is deduced from p_0 and the initial entropy distribution. In the usual case, the entropy will be uniform so that $p_0 \propto \rho_0^\gamma$.

Turning now to the shock relations, the general form simplifies in this case since the magnetic field has no normal component. The appropriate relations giving the values behind the shock in terms of the shock velocity U and the equilibrium values are

$$\mathbf{H}(U-u) = \mathbf{H}_0 U,$$

$$\rho(U-u) = \rho_0 U,$$

$$p + \tfrac{1}{2}\mu H^2 + \rho(U-u)^2 = p_0 + \tfrac{1}{2}\mu H_0^2 + \rho_0 U^2,$$

$$\tfrac{1}{2}(U-u)^2 + \frac{\gamma p}{(\gamma-1)\rho} + \frac{\mu H^2}{\rho} = \tfrac{1}{2}U^2 + \frac{\gamma p_0}{(\gamma-1)\rho_0} + \frac{\mu H_0^2}{\rho_0}.$$

It is convenient to be able to solve these relations and express all the quantities in terms of a single parameter characterizing the shock strength. This can be done in terms of the parameter $\xi = \rho/\rho_0$. Then,

$$\left.\begin{aligned} &\rho = \rho_0\xi, \qquad \mathbf{H} = \mathbf{H}_0\xi, \qquad u = U\frac{\xi-1}{\xi},\\ &U^2 = \frac{2\xi}{(\gamma+1)-(\gamma-1)\xi}\left[a_0^2 + b_0^2\left\{\left(1-\frac{\gamma}{2}\right)\xi + \frac{\gamma}{2}\right\}\right],\\ &p = p_0 + \frac{2\rho_0(\xi-1)}{(\gamma+1)-(\gamma-1)\xi}\left[a_0^2 + \frac{\gamma-1}{4}b_0^2(\xi-1)^2\right], \end{aligned}\right\} \tag{40}$$

where a_0 is the sound speed $\sqrt{(\gamma p_0/\rho_0)}$ and b_0 is the Alfvén speed $\sqrt{(\mu H_0^2/\rho_0)}$ (i.e. the speed of magnetic waves in an incompressible fluid). There are two possibilities leading to strong shocks, i.e. large values of p/p_0; either ξ is close to the value $(\gamma+1)/(\gamma-1)$ or b_0^2/a_0^2 is large. The former is the usual case found in gas dynamics, but it is an interesting fact that in magnetohydrodynamics strong shocks may also arise when the magnetic field is very strong for any value of $\xi > 1$. When $\xi \to (\gamma+1)/(\gamma-1)$, the shock relations take the simple form

$$\left.\begin{aligned} \rho &= \frac{\gamma+1}{\gamma-1}\rho_0, & \mathbf{H} &= \frac{\gamma+1}{\gamma-1}\mathbf{H}_0, \\ p &\sim \frac{2\rho_0}{\gamma+1}U^2, & u &\sim \frac{2}{\gamma+1}U. \end{aligned}\right\} \tag{41}$$

It should be noted that in this limiting case, the expressions for p, ρ, u, in terms of U are independent of the magnetic field.

Now we apply the rule described in the earlier sections to determine the motion of a converging shock. The characteristic equations corresponding to the first pair in (6) are

$$\left(\frac{1}{u\pm c}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)(p+\tfrac{1}{2}\mu H_\theta^2+\tfrac{1}{2}\mu H_z^2)\pm\rho c\left(\frac{1}{u\pm c}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)u+ \\ +\frac{\rho c^2 u}{u\pm c}\frac{1}{r}-\frac{\mu H_\theta^2}{r}\frac{u\mp c}{u\pm c}=0 \tag{42}$$

where

$$c^2 = a^2+b^2 = \frac{\gamma p}{\rho}+\frac{\mu H^2}{\rho}. \tag{43}$$

In this section the particle velocity u is taken as positive in the direction of increasing r. For a converging shock, therefore, u is negative and we must use the relation for a negative characteristic to determine the shock, i.e.

$$dp+\mu H_\theta\, dH_\theta+\mu H_z\, dH_z-\rho c\, du+\left\{\frac{\rho c^2 u}{u-c}-\frac{\mu H_\theta^2(c+u)}{u-c}\right\}\frac{dr}{r}=0. \tag{44}$$

The final step is to substitute the shock conditions (40) into this relation. A first-order equation for $\xi(r)$ is obtained, and this determines the shock.

In general, the equation will require numerical integration. But in the special case of very strong shocks with $\xi \to (\gamma+1)/(\gamma-1)$, the simplified shock conditions noted in (41) may be used. From these conditions we see that the corresponding approximations in (44) reduce it to

$$dp-\rho a\, du+\frac{\rho a^2 u}{u-a}\frac{dr}{r}=0,$$

since the other terms involving the magnetic field are of smaller order in the shock velocity U. Now, when (41) is substituted in this equation, the resulting equation for $U(r)$ integrates immediately to

$$U \propto \rho_0^{-\beta} r^{-n}, \tag{45}$$

where

$$\beta = \left\{2+\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)}\right\}^{-1}, \quad n = \gamma\left\{1+\sqrt{\left(\frac{\gamma(\gamma-1)}{2}\right)}\right\}^{-1}\left\{2+\sqrt{\left(\frac{2\gamma}{\gamma-1}\right)}\right\}^{-1}. \tag{46}$$

The corresponding law for the pressure behind the shock is

$$p \propto \rho_0^{1-2\beta} r^{-2n}. \tag{47}$$

The exponent n is exactly the one appearing in the solution for a converging cylindrical shock in a uniform medium (see equation (11)) and β is the exponent that was found in (33) for a plane shock in a non-uniform medium; (45) shows the combined effect. It is interesting to note that for a strong shock the magnetic field enters the law of propagation only indirectly through its control of the equilibrium density distribution ρ_0. This is exactly analogous to the situation found in the last section for the effect of the body force F. No doubt (45) and (47) apply in general to any case when the distribution is not uniform whatever mechanism maintains the density distribution. But for weaker shocks, the details of the particular force field would be required. Since $1-2\beta > 0$, a general consequence of (47) is that the shock will be relatively strengthened or weakened according as the density ρ_0 increases or decreases towards the centre.

The law of propagation of the shock wave can be given explicitly in the case of weak shock waves also. This case is less interesting and only the final result is noted. It is found that

$$\frac{p-p_0}{p_0} \propto \rho_0^{-1/2} c_0^{-3/2} r^{-1/2}.$$

8. Shear layers in supersonic flow

The problems in steady supersonic flow are analogous in many ways to the unsteady problems considered so far. There is, however, one important question that arises in steady flow problems; the flow downstream of the shock may be either supersonic or subsonic. The questions and methods discussed in this paper apply where the flow is supersonic; if the flow downstream of the shock is subsonic the methods are meaningless. With this restriction the problems can be solved in the same way and the perturbation methods etc. discussed earlier go through and give similar results. Accordingly the discussion will be brief.

In this section we note how Moeckel's original results can be obtained by application of the characteristic relation. We consider the plane flow represented in figure 3. The incident flow is uniform in $y < 0$ but varies with y in $y > 0$. The only requirement is that the pressure p_0 be constant; then, the density distribution $\rho_0(y)$ and velocity distribution $q_0(y)$ can be arbitrary. (In reality, of course, viscous forces will arise in such a shear layer but we must confine attention to the case when these can be neglected.)

In supersonic flow the derivation of the characteristic relations from the usual equations of motion involves much more manipulation than in the unsteady flow discussed in § 3. The details are given in most books on compressible flow (although not all books give the general non-isentropic

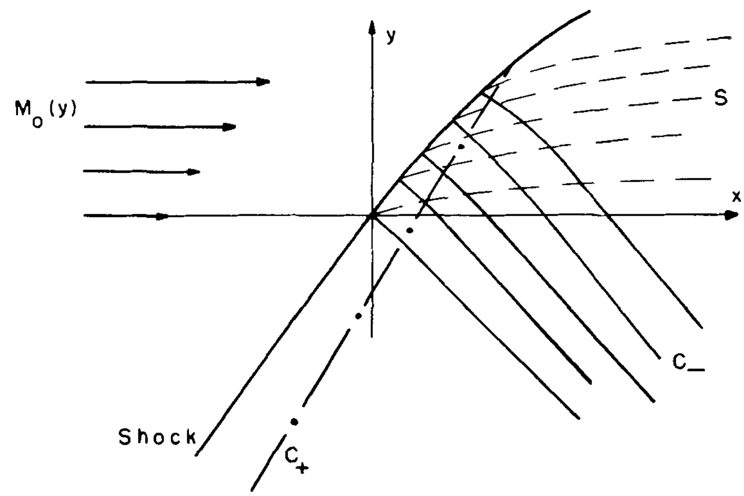

Figure 3. Interaction of an oblique shock with a shear layer.

form required here) and it will be sufficient to quote the result. On a positive characteristic (C_+ in figure 3), the flow quantities must satisfy the relation (see, for example, Howarth (1953))

$$\frac{1}{\gamma}\frac{dp}{p}+\frac{d\theta}{\cos\mu\sin\mu}=0 \quad \text{on} \quad \frac{dy}{dx}=\tan(\mu+\theta), \tag{48}$$

where p is the pressure, θ is the angle of the velocity vector to the x-axis, μ is the Mach angle $\sin^{-1}(a/q)$ where a is the sound speed $\sqrt{(\gamma p/\rho)}$. The rule is to apply this relation between p and θ along the shock.

The shock conditions may be written

$$\left.\begin{aligned} p &= p_0\left\{\frac{2\gamma M_0^2}{\gamma+1}\sin^2\beta-\frac{\gamma-1}{\gamma+1}\right\},\\ \rho &= \rho_0\left(\frac{\gamma-1}{\gamma+1}+\frac{2}{\gamma+1}\frac{1}{M_0^2\sin^2\beta}\right)^{-1},\\ \tan(\beta-\theta) &= \frac{\gamma-1}{\gamma+1}\tan\beta+\frac{2}{\gamma+1}\frac{1}{M_0^2\sin\beta\cos\beta},\\ q\cos(\beta-\theta) &= q_0\cos\beta, \end{aligned}\right\} \tag{49}$$

where β is the angle of inclination of the shock to the x-axis. From (49) all the flow quantities can be expressed in terms of β and the upstream flow. The shape of the shock is found by substituting these expressions into (48). A first-order equation is obtained for β as a function of y. Of course, any of the flow quantities can be chosen as the variable instead of β and a similar equation obtained.

Moeckel introduces the following notation for the solutions of the relations (49)

$$\left.\begin{aligned} p &= p_0 f_1(M_0, \beta), \\ \theta &= f_2(M_0, \beta), \\ \gamma(\sin\mu\cos\mu)^{-1} &= f_3(M_0, \beta). \end{aligned}\right\} \tag{50}$$

When these are substituted in (48) (noting that p_0 is constant), we have

$$\frac{d\beta}{dM_0} = -\left(\frac{1}{f_1}\frac{\partial f_1}{\partial M_0} + f_3\frac{\partial f_2}{\partial M_0}\right)\bigg/\left(\frac{1}{f_1}\frac{\partial f_1}{\partial \beta} + f_3\frac{\partial f_2}{\partial \beta}\right). \tag{51}$$

This is Moeckel's result.

9. Axisymmetrical duct in supersonic flow

An interesting application in supersonic flow is to the shock formed in the entry of an axisymmetrical duct. If r measures distance from the axis, the appropriate characteristic relation is now

$$\frac{1}{\gamma}\frac{dp}{p} - \frac{d\theta}{\cos\mu\sin\mu} + \frac{\sin\theta}{\cos\mu\sin(\mu-\theta)}\frac{dr}{r} = 0. \tag{52}$$

The change in sign from (48) arises because the shock *inside* the duct is formed by the negative characteristics; since θ is negative in this case the signs of (48) and (52) eventually agree and the only essential difference is the additional term in (52) due to the cylindrical geometry. The shock shape is then obtained by substituting the shock conditions (49) into (52). In this case we assume uniform flow upstream of the shock so that M_0 etc. are constants. Then, the equation for $\beta(r)$ takes the form

$$r\frac{d\beta}{dr} = -f(\beta), \tag{53}$$

where $f(\beta)$ is a known function. It is found that $f(\beta) > 0$ so that β increases as r decreases towards the axis; hence the strength of the shock increases towards the axis. However at some point before the axis is reached the flow behind the shock becomes subsonic and certainly (53) does not apply beyond this point; in fact, $f(\beta)$ becomes imaginary. In the real situation, Mach reflection occurs and the flow pattern becomes very complicated. It is easy to see that the Mach reflection must occur somewhere before this 'sonic point' is reached because the reflected shock must have supersonic flow upstream of it. However it seems reasonable to assume that (53) applies until the triple point is reached.

If we denote the radius of the duct by r_0 and the initial angle of the shock by β_0, (53) gives

$$\log\frac{r_0}{r} = \int_{\beta_0}^{\beta} f(\beta)\,d\beta. \tag{54}$$

The initial shock angle β_0 is a function of the initial angle θ_0 of the wall of the duct (it is given by the third equation in (49)). Thus, according to (54), *only the initial angle of the wall duct affects the shape of the shock appreciably.* This corresponds to the result for cylindrical shocks that the motion of the shock is nearly independent of the piston motion (see § 1). At the sonic point, β takes a value β_s depending only on M_0 and γ, and the position of this point is given by $r = r_s$ where

$$\log \frac{r_0}{r_s} = \int_{\beta_0(\theta_0)}^{\beta_s} f(\beta)\, d\beta. \tag{55}$$

As a check on the present method, the predictions of (55) can be compared with calculations of the shock shape by the numerical method of characteristics (Ferri 1946). In particular, it is convenient to check the dependence of r_0/r_s on θ_0. The integral in (55) was calculated for a few values of θ_0 using $M_0 = 1{\cdot}6$, the value chosen in Ferri's calculations, and the results are given in the following table.

θ_0	14·24°	11·62°	8·66°	3·59°
r_s/r_0	1·000	0·659	0·337	0·048

A graph of the function is given in Ferri's paper and the values given above fit almost on the curve; the curve is shown in figure 4. This problem is a fairly severe test of the simple rule presented here and the very close agreement with accurate calculations gives ample support of the practical usefulness of this method.

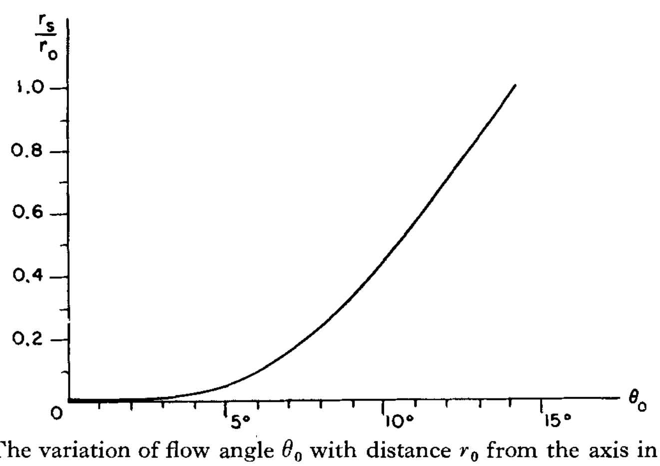

Figure 4. The variation of flow angle θ_0 with distance r_0 from the axis in supersonic flow inside a duct.

10. Propagation of bores in shallow water

The application to this case is quite straightforward but the results require special comment.

If $h(x, t)$ denotes the depth of the water, $h_0(x)$ the undisturbed value ahead of the bore, and $u(x, t)$ the particle velocity, the equations of the shallow-water theory are

$$\left.\begin{aligned} \eta_t+\{(h_0+\eta)u\}_x &= 0, \\ u_t+uu_x+g\eta_x &= 0, \end{aligned}\right\} \tag{56}$$

where $\eta = h-h_0$. The appropriate characteristic relation is

$$du+2dc = \frac{g\,dh_0}{u+c}, \tag{57}$$

where $c = \surd(gh)$. The bore conditions are

$$U = \sqrt{\left(\frac{gh(h_0+h)}{2h_0}\right)}, \tag{58}$$

$$u = \frac{h-h_0}{h}\,U. \tag{59}$$

It is convenient to introduce $M = U/\surd(gh)$ and determine first the variation of M with x. Then the height of the bore is given by

$$\eta = h-h_0 = 2h_0(M^2-1) \tag{60}$$

and the bore velocity by

$$\frac{U}{\surd(gh_0)} = M\,\surd(2M^2-1). \tag{61}$$

From (59), (60), and (61),

$$\left.\begin{aligned} \frac{c}{\surd(gh_0)} &= \surd(2M^2-1), \\ \frac{u}{\surd(gh_0)} &= \frac{2M(M^2-1)}{\surd(2M^2-1)}. \end{aligned}\right\} \tag{62}$$

Substituting these expressions in (57), we obtain

$$\frac{1}{h_0}\frac{dh_0}{dM} = -\frac{2(2M^3+2M^2-2M-1)(4M^4+4M^3-3M^2-2M+1)}{(M^2-1)(2M^2-1)(2M^4+6M^3+2M^2-3M-2)}. \tag{63}$$

The range of variation of M is $1 < M < \infty$ and in this range all the factors in (63) are positive. Therefore, M always increases as h_0 decreases and *vice versa*. For weak bores ($\eta/h_0 \ll 1$), M is near 1 so the asymptotic form for (63) is

$$\frac{1}{h_0}\frac{dh_0}{dM} = -\frac{4}{5}\frac{1}{M-1}.$$

Hence,

$$M-1 \propto h_0^{-5/4}, \qquad \eta \propto h_0^{-1/4}. \tag{64}$$

This agrees with the result of the linear theory. For strong bores ($\eta/h_0 \gg 1$), M is large and the approximate form of (63) is

$$\frac{1}{h_0}\frac{dh_0}{dM} = -\frac{4}{M}.$$

Hence

$$M \propto h_0^{-1/4}, \qquad \eta \propto h_0^{1/2}. \tag{65}$$

It should be noted that for weak bores η increases as h_0 decreases, but for strong bores η decreases as h_0 decreases. The critical value M_c of M which separates these two regimes is the value for which

$$\frac{d\eta}{dh_0} = 2(M^2-1)+4h_0 M\frac{dM}{dh_0} = 0.$$

The equation for M_c is then found from (63) and the value for M_c can be calculated. It is found that M_c is approximately 1·2; the corresponding value of η/h_0 is 0·9.

This result for the variation of η with h_0 has strange consequences when we apply the above theory to consideration of a bore coming into the shore line of a sloping beach. Assuming that the bore is sufficiently weak initially, it will increase in height as expected, but eventually, although its strength η/h_0 continues to increase, its height η will decrease. In the final stage, $\eta \propto h^{1/2}$ as given by (65) so the height of the bore tends to zero. This is certainly not in accord with preconceived notions of what should happen. We might consider that the lowest order approximation is not adequate to treat the extreme case in which $h_0 \to 0$. It would not be surprising; in the analogous problem of a shock moving into a stratified medium, Chisnell found that it was necessary to go to a second approximation (i.e. include 're-reflected waves' in his theory) in such extreme cases. However, for this water-wave problem the idealizations of the basic shallow-water theory and bore conditions are drastic ones, and we can see immediately that they lead to trouble. For, one expects that in the true situation the bore velocity U and height h approach finite values at the shore. But, if this is so (58) cannot hold as $h_0 \to 0$. Thus any theory which includes (58) as a basic condition cannot lead to the expected results. In fact (65) does predict a finite value for U so (63) gets one of the expected results. For η to have a finite value as $h_0 \to 0$, U must become infinite like $h_0^{-1/2}$ according to (58); this result would have been unsatisfactory also. It is easy to see why shallow-water theory may break down when strong bores are involved. In shallow-water theory vertical accelerations are neglected and the motion is assumed to be essentially horizontal. It is possible, therefore, that the theory breaks down when the large change in water levels at a very strong bore is considered.

On the other hand, even though the idealized model may not describe reality very well it does conserve mass, and the feeling that the height of the bore should be finite is based partly on this. For the height of the water surface must be increasing on the whole, since we are assuming that there is a region of uniform depth away from the shore providing a continual inflow of water. If the bore height tends to zero it must be followed by large (continuous) increases in depth.

To settle these points much more extensive investigations are necessary.

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